# Design of perfect reconstruction rational sampling filter banks*

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**Abstract** The design of rational sampling filter banks based on a recombination structure can be formulated as a problem with two objective functions to be optimized. A new hybrid optimization method for designing perfect-reconstruction rational sampling filter banks is presented, which can be used to solve a class of problems with two objective functions. This method is of good convergence and mezzo calculation cost. Satisfactory results free of aliasing in analysis and synthesis filters can be obtained by the proposed method.

**Keywords:** rational sampling banks, perfect reconstruction, evolutionary strategies.

In some applications such as audio coding and subband adaptive filtering, a nonuniform frequency partitioning may be preferred. Efficient structure and design procedures for general nonuniform perfect reconstruction (PR) filter banks are therefore desirable. The filter banks with rational sampling factors are nonuniform filter banks. The PR nonuniform filter bank has been studied by many researchers[1~4]. The PR condition of a nonuniform filter bank was first studied by Hoang and Vaidyanathan[5]. Cox[3] has proposed a two-stage structure for nonuniform filter bank, where certain channels in an $M$-channel uniform filter bank are recombined or merged together using the synthesis filters of another filter bank with a smaller channel number. In his work, the analysis and synthesis filter banks were derived from the pseudo-quadrature mirror filters, therefore they are not perfectly reconstructed. In 1992, the theory and design of a class of PR cosine-modulated uniform filter bank (CMFB) was proposed by Koilpillai and Vaidyanathan[6]. Based on the above study, Chan and Xie[7] designed a nonuniform PR filter bank by using CMFB on Cox's two-stage structure.

In Cox's two-stage structure, an $M$-channel uniform filter bank called the source filter bank and an $m$-channel uniform filter bank called the recombining filter bank ($m < M$) are required to design reseparately. The frequency responses of the two filter banks must be carefully designed otherwise protrusions in stopband of analysis (synthesis) filters will appear[7]. In Ref. [7], to suppress the protrusions and obtain a good stopband attenuation, the stopband cutoff frequencies of two prototype filters in the uniform CMFB need to be adjusted, which increases the complexity of design, and only the case of $m$ and $M$ being coprime is discussed. The case of $m$ and $M$ being non-coprime is discussed and the design of nonuniform cosine-modulated filter banks is formulated as a problem with two objective functions to be optimized based on the two-stage structure in this paper. And a new

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hybrid optimization method for designing PR nonuniform filter banks is presented, which can be used to solve a class of problems with two objective functions. With this method, the protrusions in stopband of analysis (synthesis) filters of the nonuniform filter bank can be suppressed.

## 1 Cosine-modulated nonuniform filter bank

In the $M$-channel CMFB, the analysis and synthesis filter banks $h_k(n)$ (or $H_k(z)$) and $f_k(n)$ (or $F_k(z)$) are obtained[6] by cosinoidally modulated the low pass prototype filter $p(n)$ as follows

$$h_k(n) = p(n)\cos\left[\frac{\pi}{2M}(2k+1)\left(n-\frac{N-1}{2}\right)+(-1)^k\frac{\pi}{4}\right], \tag{1}$$

$$f_k(n) = p(n)\cos\left[\frac{\pi}{2M}(2k+1)\left(n-\frac{N-1}{2}\right)-(-1)^k\frac{\pi}{4}\right], \tag{2}$$

where $k=0,1,\cdots,M-1$, $n=0,1,\cdots,N-1$, and $N$ is the length of filter $p(n)$. Let $P(z) = \sum_{n=0}^{N-1} p(n)z^{-n} = \sum_{q=0}^{2M-1} z^{-q}E_q(z^{2M})$ be the type-I polyphase decomposition[8] of the prototype filter. If $N=2mM$, the length of each $E_k(z)$ will be $m$. The PR conditions[6] are given by

$$E_k(z)E_{2M-k-1}(z) + E_{M+k}(z)E_{M-k-1}(z) = \beta\cdot z^{-\alpha}, \qquad k=0,1,\cdots,M-1 \tag{3}$$

where $\beta$ is a certain constant, $\alpha\,(\alpha<2m-1)$ is some integer. Since the frequency responses of $h_k(n)$ and $f_k(n)$ are determined by the prototype filter $P(z)$, it is only necessary to minimize $P(z)$ in the stopband. The design problem can be formulated as the following constrained optimization

$$\min_p \Phi_{\text{CMFB}} = \min_p\left(\int_{\omega_S}^{\pi} |P(e^{j\omega})|^2 d\omega\right) \tag{4}$$

subjected to the PR constraints in Eq. (3) for CMFB. The value of the cutoff frequency $\omega_S$ depends on the desired transition bandwidth. It should be between $\pi/2M$ and $\pi/M$.

The two-stage structure of Cox is shown in Fig. 1. As described in Ref. [7], the first $m_k$ channels of analysis filter in the $M$-channel uniform filter bank $H_i(z)$ $(i=0,\cdots,m_k-1)$ are merged using the synthesis filters of an $m_k$-channel uniform filter bank $G_{o,i}(z)$. The analysis section $G'_{o,i}(z)$ is used to produce the $m_k$ subbands and are merged by the synthesis filters $F_i(z)$ of the $M$-channel uniform filter bank. The sampling rate after merging is reduced by a factor of $m_k/M$. So the first channel of nonuniform filter banks can be obtained. The rest can be deduced by analogy, so $L$-channel nonuniform filter banks with sampling rates $\left\{\frac{m_0}{M},\cdots,\frac{m_k}{M},\cdots,\frac{m_{L-1}}{M}\right\}$ can be obtained. For the critical sampling, they have to satisfy the condition $\sum_{k=0}^{L-1} m_k = M$. It can be seen that if $G_{o,i}(z)$ and $G'_{o,i}(z)$ form an $m_k$-channel perfect reconstruction filter bank, $z^{-1}G_{o,i}(z)$ and $G'_{o,i}(z)$, which are enclosed with dotted lines in Fig. 1, constitute a perfect reconstruction transmultiplexer[8]. If the delay, which is introduced by the transmultiplexes, is compensated in the each branches of the $M$-channel filter bank, the entire system is PR. The source and recombination filter banks can be designed by the CMFB. If they are designed independently, a protrusion in stopband of analysis (synthesis) filters of the nonuniform filter will emerge. This is not allowed in the application of the filter banks. An optimization method to solve this problem will be discussed in the next section.

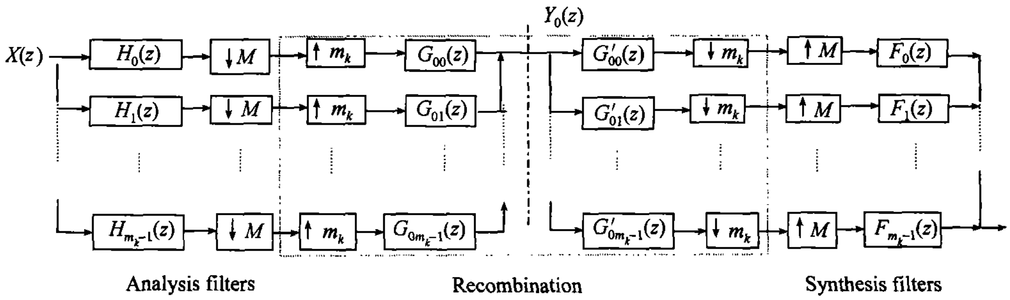


Fig. 1 Structure of Recombination Nonuniform PR filter bank (only the first $m_k$ ($k = 0$) branches are plotted).

## 2 Objective function

The objective function to be optimized is dependent on $M$ is coprime or non-coprime to $m_k$.

### 2.1 Case for $M$ coprime to $m_k$

If $M$ is coprime to $m_k$, the decimators and the interpolators can be interchanged (Fig. 1). Using the Noble identity[9], an equivalent filter $\widetilde{H}_k(z)$ is obtained[7]

$$\widetilde{H}_k(z) = \sum_{i=0}^{m_k-1} H_{l+i}(z^{m_k}) G_{ki}(z^M), \tag{5}$$

where $G_{ki}(z)$ is the $i$-th synthesis filter in the $m_k$-channel uniform filter bank for $k$-band nonuniform filter bank ($0 \leqslant k \leqslant L-1$), and $l$ ($0 \leqslant l \leqslant M-1$) is the starting index of merging. Therefore, an appropriate measure of the sum of stopband energy of $\widetilde{H}_k(z)$ is chosen as an objective function to be minimized as follows.

$$\min(\Phi_k) = \min\left(\int_0^{\tilde{\omega}_{s1}} |\widetilde{H}_k(e^{j\omega})|^2 d\omega + \int_{\tilde{\omega}_{s2}}^{\pi} |\widetilde{H}_k(e^{j\omega})|^2 d\omega\right), \tag{6}$$

where $\tilde{\omega}_{s1}$ and $\tilde{\omega}_{s2}$ are low and high cutoff frequencies of stopband of the equivalent filter $\widetilde{H}_k(z)$, respectively, which can be chosen the following equations.

$$\begin{cases} 0 = \tilde{\omega}_{s1} < \tilde{\omega}_{s2} < \pi, & \text{if } k = 0, \\ 0 < \tilde{\omega}_{s1} < \tilde{\omega}_{s2} = \pi, & \text{if } k = L-1, \\ 0 < \tilde{\omega}_{s1} < \tilde{\omega}_{s2} < \pi, & \text{if } 0 < k < L-1. \end{cases} \tag{7}$$

### 2.2 Case for $M$ non-coprime to $m_k$

For this case, the decimators and the interpolators can not be interchanged (Fig.1), so there is no equivalent filter representation. The relation between input $X(z)$ and output $Y_k(z)$ of the nonuniform analysis filter can be expressed as

$$Y_k(z) = \frac{1}{M}\sum_{i=0}^{m_k-1} G_{ki}(z)H_{l+i}(z^{m_k/M})X(z^{m_k/M}) + \frac{1}{M}\sum_{i=0}^{m_k-1} G_{ki}(z)\sum_{s=1}^{M-1} H_{l+i}(z^{m_k/M}W_M^s)X(z^{m_k/M}W_L^s), \tag{8}$$

where $W_M = e^{j2\pi/M}$. The first term in Eq. (8) is the desired components and the second term is the aliasing components. If there exists only one component with frequency $\omega_1$ in input signal $X(z)$, $Y_k(z)$ will consist of desired frequency $\omega_1 \times \frac{m_k}{M}$ and other harmonic components. From Eq. (8),

$$Y_k(z) = \frac{1}{M}\sum_{i=0}^{m_k-1} G_{ki}(z)H_{l+i}(z^{m_k/M})X(z^{m_k/M}) + \frac{1}{M}\sum_{s=1}^{M-1} A_k(z)X(z^{m_k/M}W_M^s), \tag{9}$$

$$A_k(z) = \frac{1}{M}\sum_{i=0}^{m_k-1} G_{ki}(z)H_{l+i}(z^{m_k/M}W_M^s). \tag{10}$$

If the gain $A_k(z)$ of the aliasing components $X(z^{m_k/M}W_M^s)$ is very small, the harmonic components in $Y_k(z)$ will be very small. Therefore an appropriate objective function should be chosen as follows

$$\min(\Phi_k) = \min\left(\int_0^{\pi}\sum_{s=1}^{M-1}\left|A_k(e^{j\omega})\right|d\omega = \frac{1}{M}\sum_{s=1}^{M-1}\int_0^{\pi}\left|\sum_{i=0}^{m_k-1} G_{ki}(e^{j\omega})H_{l+i}(e^{jm_k/M\omega}W_M^s)\right|d\omega\right). \tag{11}$$

## 3 Variables to be optimized

We find that the $\omega_s$ cutoff frequency of stopband in the prototype filter, $\omega_s$ will directly influence the shape of transition band of filters in the uniform filter bank. From Eqs. (5) and (8), the filter quality of the analysis filters in the nonuniform filter bank depends on the frequency responses of $G_{ki}(z)$ and $H_{l+i}(z)$. Let us denote by $N_h$ the filter length of the prototype filter $P_h(z)$ of $M$-channel uniform source filter bank, and by $N_g$ that of the prototype $P_g(z)$ of $m_k$-channel the recombination filter bank. Their cutoff frequencies of stopband are $\frac{\pi}{2M}$ and $\frac{\pi}{2m_k}$ respectively in theory. Note that after downsampling $M$ and upsampling $m_k$, the frequency response of $H_{l+i}(e^{j\omega})$ will be compressed into the frequency response of $H_{l+i}(e^{j\frac{m_k-2S\pi}{M}\omega})$ ($S = 0, 1, 2\cdots$). The protrusions in stopband of $H_{l+i}(e^{j\frac{m_k-2S\pi}{M}\omega})G_{ki}(e^{j\omega})$ ($i = 0, m_k - 1$) are caused by the aliasing components from the decimators and the compression of the frequency responses from interpolators. The more similar the shape of the transition band of $P_h(e^{j\frac{m_k}{M}\omega})$ is to that of $P_g(e^{j\omega})$[3], the smaller the summing up $\sum_{i=0}^{m_k-1} H_{l+i}(e^{j\frac{m_k-2S\pi}{M}\omega})G_{ki}(e^{j\omega})$ protrusions and the harmonic components in $Y_k(e^{j\omega})$ are. Two real factors $\varepsilon_1(0.5 \leqslant \varepsilon_1 \leqslant 1.5)$ and $\varepsilon_2(0.5 \leqslant \varepsilon_2 \leqslant 1.5)$ are introduced to express the cutoff frequencies of the prototype filters as $\omega_{s\_h} = \frac{\pi}{2M} \times \varepsilon_1$ and $\omega_{s\_g} = \frac{\pi}{2m_k} \times \varepsilon_2$[10], respectively. It is possible that the transition bands of $P_h(e^{j\frac{m_k}{M}\omega})$ is similar to that of $P_g(e^{j\omega})$ if the lengths of two prototype filters satisfy the following condition given by Cox[13].

$$N_h = 2 \times n \times M \quad \text{and} \quad N_g = 2 \times n \times m_k, \tag{12}$$

where $n$ is a non-zero positive integer. The design procedures of the PR nonuniform filter banks based on the two-stage structure are summarized as follows:

(i) Choose the lengths of prototype filters by Eq. (12) and factor $\varepsilon_1$.

(ii) Design an $M$-channel PR uniform CMFB.

(iii) Design the coefficients of $P_g(z)$ and factor $\varepsilon_2$ in order to minimize the objective functions, $\Phi_{MFB\_g}$ and $\Phi_k$, where

$$\Phi_{CMFB\_g} = \int_{\omega_{s\_g}}^{\pi} |P_g(e^{j\omega})|^2 d\omega. \tag{13}$$

Hence, the variables to be optimized are the coefficients of $P_g(z)$ and factor $\varepsilon_2$.

## 4 Algorithm of the hybrid optimization

The problem with two objective functions cannot be solved by a traditional optimization method. A hybrid optimization method consisting of the constrained successive quadratic programming algorithm (SQP) and the evolutionary strategies (ES) algorithm is proposed to solve the problem. The SQP is used to minimize the objective function $\Phi_{CMFB\_g}$ with the coefficients of $P_g(z)$ as variables. The ES algorithm is used to minimize the objective function $\Phi_k$ in Eq. (6) or Eq. (11) with variable $\varepsilon_2$. There are many possible solutions for $\varepsilon_2$ in a generation of evolutionary strategies. For all possible solutions of $\varepsilon_2$, the optimal prototype filters $P_g(z)$ are designed by SQP, then the values of objective function $\Phi_k$ based on the optimal prototype filters can be calculated. The best prototype filters $P_g(z)$ can be found in this generation of ES. The variable $\varepsilon_2$ will be changed according to ES so that the next generation searching can be finished. In each generation of ES, an optimal $\varepsilon_2$ is always kept to guarantee the convergence of the hybrid optimization method. Since there is only a variable $\varepsilon_2$ in ES, the calculation complexity of the method is mezzo. The algorithm of this hybrid method is described as follows.

(i) Initialization of parent population. A uniform distribution of $\varepsilon_{2i}$ $(i = 1, 2, \cdots, P)$ is generated randomly in $[0.5, 1.5]$. $P$ is the size of the population.

(ii) Copying. The new generation $\varepsilon'_{2i}$ can be obtained by adding a Gauss's random noise with zero mean and some variance to each $\varepsilon_{2i}$ in parent population.

(iii) Second optimization. For all $\varepsilon_{2i}$ and $\varepsilon'_{2i}$, using SQP to minimize the objective function $\Phi_{CMFB\_g}$ subjected to the PR constraints in (3), the optimal coefficients of $p_{g_i}(n)$ and $p'_{g_i}(n)$ $(i = 1, 2, \cdots, p))$ can be obtained.

(iv) Selection. Through calculating the objective function $\Phi_k$ based on $p_{g_i}(n)$ and $p'_{g_i}(n)$, the new parent generation population $\varepsilon_{2i}(i=1,2,\cdots,P)$ can be found, which makes the objective function $\Phi(\varepsilon_{2i}, p_{g_i}(n))$ and $\Phi(\varepsilon'_{2i}, p'_{g_i}(n))$ smaller.

(v) Iteration. The process the above Steps (ii) ~ (iv) are repeated till the optimal solution is found or the maximum iteration generation is over.

## 5 Examples and conclusion

Two nonuniform PR filter banks as examples are designed using the proposed method. The first example is a 2-channel nonuniform PR filter bank with rational sampling factors $\left\{\frac{3}{4}, \frac{1}{4}\right\}$, where $M$ is coprime to $m_k$. The frequency responses of the equivalent analysis filters are shown in Fig. 2. The second example is a 3-channel nonuniform PR filter bank with rational sampling factors $\left\{\frac{2}{4}, \frac{1}{4}, \frac{1}{4}\right\}$, where in the first channel $M$ is non-coprime to $m_k$. For the two examples, three uniform PR CMFBs with 2, 3 and 4-channel number are designed. The lengths and factors $\varepsilon$ of their prototype filters $(p_2, p_3, p_4)$ are listed in Table 1. The frequency responses of analysis filters of uniform 4-channel CMFB are shown in Fig. 3.

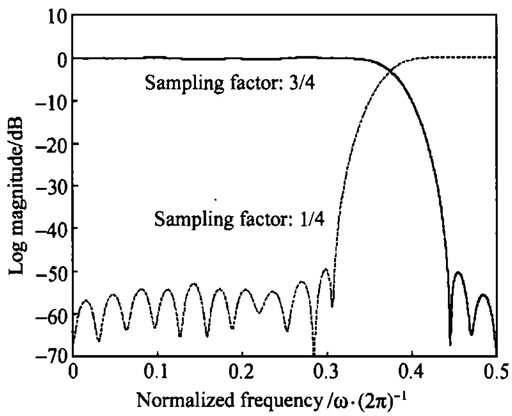


Fig. 2 Frequency responses of the equivalent analysis filters with sampling factors (3/4, 1/4).

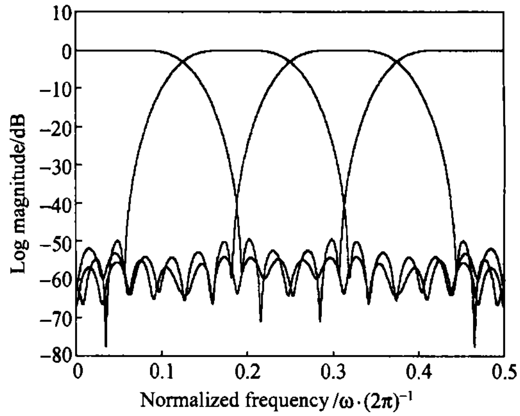


Fig. 3 Magnitude responses for the 4-channel uniform CMFB.

Table 1 Lengths of prototype filters and factors $\varepsilon$

| Prototype filter | Length | $\varepsilon$ |
|---|---|---|
| $p_2$ | 20 | 1.00064 |
| $p_3$ | 30 | 1.00750 |
| $p_4$ | 40 | 1.00000 |

From the results, stopband attenuation of the equivalent filters in Fig. 2 is low over − 50 dB that the proposed method is effective and feasible.

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